JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 21, 547-555 (1968)

Properties of Solutions of Laplace's Equation Generated by Integrals in the Complex Domain I. Axially Symmetric Potentials

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Submitted by R. Duffin

The purpose of this paper is to set up classes of axially symmetric potential functions (solutions of Laplace's equation) for which some of the equipotential surfaces have discontinuities. These functions would be applicable to the solution of boundary value problems with discontinuous boundaries. One example of such a boundary is a plane with a hole. The potential is constant at all points except for the hole over which it is not prescribed. Another example is an open surface of finite dimensions, such as a hollow hemisphere at constant potential. These boundaries occur in physical problems involving conducting surfaces with holes or forming bowl-like structures. The functions are derived from a generating function having the form of an integral in the complex domain. In this paper, intended as the first of a series, we restrict ourselves to axial symmetry.

The present approach was suggested by the following transformation in the complex plane:

$$W(z) = U(x, y) + iV(x, y) = \sqrt{z^2 - a^2}, \; a \quad 0, \quad \text{constant.}$$

The feature of this transformation that appeared susceptible of generalization is the vanishing of either the imaginary or the real part of the complex potential depending on the values of the variables.

$$W(x, 0) = U(x, 0) + iV(x, 0) = \sqrt{x^2 - a^2}.$$

When

$$x^2 \geqslant a^2, \qquad U = \sqrt{x^2 - a^2}, \qquad V = 0;$$

when

$$x^2 \leqslant a^2, \qquad U = 0, \qquad V = \sqrt{a^2 - x^2}.$$

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Let $V(x, y)$ be the function of interest. The expressions above show that it vanishes at the boundary $y = 0$ except for the interval $-a < x < +a$. Thus it may be considered the solution of the boundary value problem: $V = 0$ at $y = 0$ except for the interval $-a < x < +a$, for which no value is prescribed.

The general solution of Laplace's equation in cylindrical coordinates which is symmetrical about the axis is known to have the form [1]

$$W(z, r) = \int_0^\pi F(z + ir \cos \omega) \frac{d\omega}{\pi}. \tag{1}$$

The author has previously shown [2–4] that a general solution can also be written as a definite integral between arbitrary limits, which are complex constants, i.e., independent of z and r.

$$W(z, r) = U(z, r) + iV(z, r) = \int_{a_1+ib_1}^{a_2+ib_2} \frac{f(t)\, dt}{\sqrt{(t - z)^2 + r^2}}. \tag{2}$$

Equation (2) is seen to differ from the Poisson formula as given, e.g., by Heins [5].

The relation between the functions f and F is given by

$$W(z, 0) = F(z) = \int_{a_1+ib_1}^{a_2+ib_2} \frac{f(t)\, dt}{\sqrt{(t - z)^2}}.$$

No convergence problem arises and a solution exists for either branch of the square root. The two solutions correspond to different branches of a Riemann surface.

When we compare the expressions for $W(z, r)$ in terms of the kernels F and f, we obtain

$$W(z, r) = \int_0^\pi F(z + ir \cos \omega) \frac{d\omega}{\pi} = \int_{\cos^{-1}(-iA_2)}^{\cos^{-1}(-iA_1)} f(z + ir \cos \omega)\, d\omega, \tag{3}$$

$$A_j = \frac{a_j - z + ib_j}{r}, \qquad j = 1, 2. \tag{3}$$

(The subscript j will have this meaning throughout this paper.)

Solutions corresponding to simple forms for the kernel F are well known [1]. We propose to study the functions generated by simple expressions for the kernel f. If $f(t)$ has a converging Laurent series expansion in t, integration in Eq. (2) can be readily performed. The main problem is to separate the real and imaginary parts $U(z, r)$ and $V(z, r)$, which are both solutions of Laplace's equation. Since these functions are obtained in closed form, any discontinuities in the equipotential surfaces become readily apparent. For example, for any given constant C, the equation $V(z, r) = C$ may not hold for every (real) value of z.

We shall study the properties of $W(z, r)$ when $f = 1$.

$$W(z, r) = \sinh^{-1} A_2 - \sinh^{-1} A_1 . \tag{4}$$

We specify the problem by setting

$$a_2 > a_1 > 0, \qquad b_2 > b_1 > 0 \tag{5}$$

and define U_j and V_j by

$$\sinh (U_j + iV_j) = A_j \tag{6}$$

so that

$$\sinh U_j \cos V_j = \frac{a_j - z}{r}, \qquad \cosh U_j \sin V_j = \frac{b_j}{r}. \tag{7}$$

Equations (7) define two different branches of a Riemann surface, in each of which U_j and V_j are uniquely defined. The expressions for $\cot V_j$ and $\tanh U_j$ in terms of z and r represent elliptic coördinates of the first type[6] (those for an oblate spheroid). When one of the limits of integration in Eq. (2) is 0, the present potential function for a kernel $f(t)$ in polynomial form can be shown to be directly related to a particular solution of Laplace's equation in elliptic coordinates.

In view of Eq. (5), Eq. (7) postulates $\sin V_j \geqslant 0$ so that $0 \leqslant V_j \leqslant \pi$. (Obviously, V_j may approach either limit of its range of variation only as $r \to \infty$.) Two special cases may arise:

(1) If U_j is always positive, then V_j may range from 0 to π.

(2) If U_j can be negative, then V_j must be restricted to the range from 0 to $\pi/2$.

The two cases correspond to different branches of a Riemann surface.

To bring into evidence the behavior of the functions U_j and V_j and the shape of the equipotential curves (i.e., of the intersections of equipotential surfaces with an axial plane), we rewrite Eq. (4) in the form

$$\sinh^2 W = A_1{}^2 + A_2{}^2 - 2A_1A_2 \cosh W. \tag{8}$$

Substituting for A_1 and A_2 their expressions given in Eq. (3) and separating real and imaginary parts, we obtain two equations each involving both U and V, which may be solved, e.g., by the Sylvester method. We show here in detail the derivation of the zero equipotential.

When $V = 0$, the real and imaginary parts of Eq. (8) become

$$r^2(\cosh^2 U - 1) = (a_2 - z)^2 + (a_1 - z)^2 - b_2^2 - b_1^2 - 2\cosh U[(a_2 - z)(a_1 - z) - b_1 b_2]$$

$$\cosh U[b_1(a_2 - z) + b_2(a_1 - z)] = b_1(a_1 - z) + (a_2 - z)\, b_2 . \tag{9}$$

Elimination of $\cosh U$ gives for the equipotential curve $V = 0$

$$r^2 = (z - z_1)(z - z_2)(z_3 - z)(z - z')^{-1}, \tag{10}$$

where

$$z' = \frac{a_1 + a_2}{2}, \qquad z_1 = \frac{b_1 a_2 + b_2 a_1}{b_1 + b_2},$$

$$2z_2 = a_1 + a_2 + \frac{b_2^2 - b_1^2}{a_2 - a_1}, \qquad z_3 = \frac{b_2 a_1 - b_1 a_2}{b_2 - b_1}. \tag{11}$$

Similarly, when $U = 0$, we obtain

$$r^2(\cos^2 V - 1) = (a_2 - z)^2 + (a_1 - z)^2 - b_2^2 - b_1^2 - 2\cos V[(a_2 - z)(a_1 - z) - b_1 b_2]$$

$$\cos V[b_1(a_2 - z) + b_2(a_1 - z)] = b_1(a_1 - z) + (a_2 - z)\, b_2 . \tag{9'}$$

Since Eqs. (9′) in $\cos V$ are identical with Eqs. (9) in $\cosh U$, the equipotential $U = 0$ is expressed by the same formula as the equipotential $V = 0$. However, there is the following distinction:

The second one of Eqs. (9) defines $\cosh U$ so that

$$\frac{b_1(a_1 - z) + b_2(a_2 - z)}{b_1(a_2 - z) + b_2(a_1 - z)} \geqslant 1, \tag{12}$$

which gives $z \leqslant z_1$ as the range of validity.

On the other hand, the second one of Eqs. (9′) gives the identical expression for $\cos V$ so that the inequality inverse to (12) must be satisfied, for which the range of validity is $z \geqslant z_1$.

To determine the general behavior of the equipotential curves in the vicinity of the axis, we apply the formula

$$V(z, r) = V(z, 0) - \frac{r^2 V''(z, 0)}{2^2}.$$

Using this expansion, we find that an equipotential curve with a single axial point is open (i.e., it is not a closed curve tangent to the axis). A curve with two axial points is closed. The open equipotential curves correspond to bowl-like equipotential surfaces while closed curves correspond to closed surfaces.

We are ready now to draw some conclusions as to the shapes of the equipotential curves.

In Eq. (10) for the zero equipotential, the right-hand side must obviously be positive. The sequence of roots of the right-hand side can readily be shown to be

$$z_3 \qquad a_1 \qquad z_1 \qquad z' \qquad z_2 \,. \tag{13}$$

Hence the equipotential can only vanish in the ranges

$$z_3 \leqslant z \leqslant z_1 \qquad \text{and} \qquad z' \leqslant z \leqslant z_2 \,. \tag{14}$$

The discussion above indicates that the first of these corresponds to $V = 0$ and the second one to $U = 0$.

In dealing with the function $V(z, r)$, Cases I and II must be considered separately.

CASE I. On the basis of Eqs. (7), at $z = a_j$, $\sin V_j = \pi/2$ for $r \leqslant b_j$ and $\sin V_j = b_j/r$ for $r > b_j$. Since in case I $\sinh U_j > 0$, $\cos V_j \gtrless 0$ for $z \lessgtr a_j$. This results in a discontinuity at $z = a_j$ extending from infinity to $r = b_j$ with V_j suffering a jump $\Delta V_j = \pi - 2\sin^{-1}(b_j/r)$ between $z = a_j + 0$ and $z = a_j - 0$. At infinite values of r, $V_j = 0$ for $z < a_j$ and $V_j = \pi$ for $z > a_j$. Thus the function V has two cuts in the (z, r) plane, at $z = a_1$ and at $z = a_2$.

While according to relations (14), the equipotential curve $V = 0$ can extend through the range $z_3 \leqslant z \leqslant z_1$, the above considerations of the functions V_1 and V_2 indicate that in the range $a_1 < z < a_2$, $V_2 < \pi/2$ while $V_1 > \pi/2$ so that $V = V_2 - V_1$ cannot vanish in this range. Therefore the equipotential curve $V = 0$ starts on the axis at $z = z_3$ and ends at $z = a_1$, $r = b_1[1 + (a_2 - a_1)^2/(b_2^2 - b_1^2)]^{1/2}$, i.e., it is an open curve. Then end point is at $z = a - 0$ on the $z = a_1$ cut. The value of the potential at $z = a_1 + 0$ is $V = 2\sin^{-1}[1 + (a_2 - a_1)^2/(b_2^2 - b_1^2)]^{-1/2} - \pi$.

Other characteristics of $V(z, r)$ are determined from the behavior of $V(z, 0)$. Calculations show that it has a positive maximum and a negative minimum. The equipotential curves for the two extreme values

$$V = \cot^{-1}\left[\frac{(a_2 - a_1)(b_2 + b_1) \pm 2S}{(b_2 - b_1)^2}\right]$$

end on the axis at $z = (b_2 a_1 - b_1 a_2 \mp S)/(b_2 - b_1)$ where

$$S^2 = b_1 b_2[(a_2 - a_1)^2 + (b_2 - b_1)^2].$$

We can conclude that in this case the function V represents the solution of the following axially symmetrical boundary value problem:

$$V = 0 \quad \text{for} \quad z = \pm\infty, \quad \text{as well as for} \quad r = \infty \quad \text{if} \quad z < a_1 \quad \text{or} \quad z > a_2$$

$$V = -\pi \quad \text{for} \quad r = \infty \quad \text{and} \quad a_1 < z < a_2 \,.$$

V assumes prescribed constant values, positive and negative, respectively, over each of two open surfaces of finite extent intersected by the axis (see Fig. 1). A surface of this type constitutes a discontinuous boundary. While the axial distance between these surfaces can be prescribed, their shape is determined by the other parameters. Therefore, an actual problem of the general type described above can use the present function as a guideline, but higher orden terms in the kernel function have to be considered for a more adequate solution.

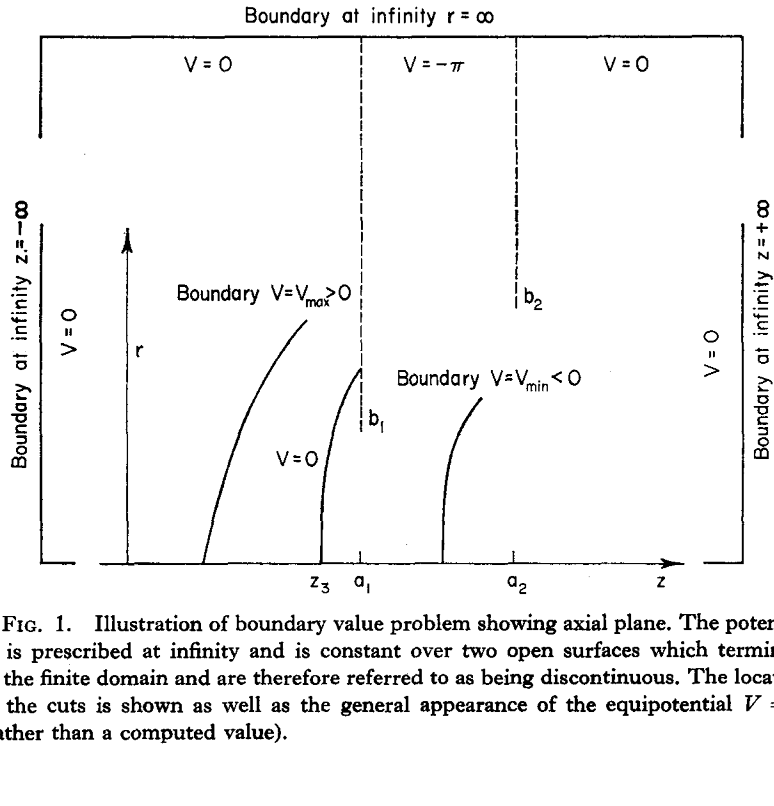

FIG. 1. Illustration of boundary value problem showing axial plane. The potential V is prescribed at infinity and is constant over two open surfaces which terminate in the finite domain and are therefore referred to as being discontinuous. The location of the cuts is shown as well as the general appearance of the equipotential $V = 0$ (rather than a computed value).

CASE II. As stated in the definition of the special cases, $V_j \leqslant \pi/2$ in the entire domain. The function $V(z, r)$ thus exhibits no cuts. Furthermore, as regards the equipotential $V(z, r) = 0$, no restrictions can arise on the range of variation of z given by the first one of relations (14). Thus according to Eq. (10), the zero equipotential curve has two axial points so that the equipotential surface is closed.

The function $V(z, 0)$ can be readily shown to have two positive unequal maxima and one negative minimum. These values constitute the maxima and the minimum, respectively, of the function $V(z, r)$. The three corresponding curves are open curves since each has only one axial point. In particular, the minimum curve $V(z, r) = -\cot^{-1}[b_2/(a_2 - a_1)]$ intersects the axis at $z = a_1$ and is entirely contained in the region between the z-axis and the curve $V(z, r) = 0$.

We can conclude that in Case II, V is the solution of an axially symmetric boundary value problem where the boundary value at infinity is 0. Two bowl-shaped surfaces of finite extent are at different positive potentials. Between them, there is an additional bowl-shaped surface at a negative potential.

A discussion of the properties of the function $U(z, r)$ in Cases I and II is omitted since the results can be obtained by reasoning analogous to that given above.

Consider now a kernel function that leads to solutions in the form of elliptic integrals. Let $f(t) = 1/\sqrt{t}$, then

$$W(z, r) = \int_{a_1+ib_1}^{a_2+ib_2} \frac{dt}{\sqrt{t[(t - z)^2 + r^2]}}. \tag{15}$$

According to formula 239 of the Handbook of Elliptic Integrals [7], Eq. (24) reduces to

$$W(z, r) = \frac{F(\Psi_2, k) - F(\Psi_1, k)}{\sqrt{p}}, \tag{16}$$

using conventional notation for elliptic functions. The parameters in Eq. (16) are related to those in Eq. (15) by

$$p^2 = z^2 + r^2, \qquad k^2 = \frac{1 + z/p}{2},$$

$$\cos \Psi_j = \cos(\sigma + i\lambda) = (p - a_j - ib_j)/(p + a_j + ib_j). \tag{17}$$

According to formula 115.01 of Ref. [7], the real and imaginary parts in Eq. (16) may be separated giving

$$\begin{aligned} W(z, r) &= U(z, r) + iV(z, r) \\ &= \frac{[F(\nu_2, k) - F(\nu_1, k)] + i[F(\mu_2, k') - F(\mu_1, k')]}{\sqrt{p}}, \end{aligned} \tag{18}$$

where

$$\tan^2 \nu_j(1 + k^2 \tan^2 \mu_j) = \tan^2 \sigma_j, \qquad \cos^2 \nu_j \cos^2 \mu_j = \frac{\cos^2 \sigma_j}{\cosh^2 \lambda_j} \tag{19}$$

Though Eqs. (19) are equivalent to the last two equations of 115.01 [7], they are given explicitly since they are much less cumbersome and permit the direct determination of ν_j and μ_j as functions of σ and λ. Thus

$$\tan^4 \nu_j + \tan^2 \nu_j \left\{1 + \frac{1}{k'^2}\left[\frac{k^2 \cosh^2 \lambda_j}{\cos^2 \sigma_j} - \tan^2 \sigma_j\right]\right\} - \tan^2 \frac{\sigma_j}{k'^2} = 0. \tag{20}$$

The determination of $\cos^2 \mu_j$ follows directly. In turn $\tan^2 \sigma_j$ and $\cosh^2 \lambda_j$ are obtained from Eqs. (17) by simple algebra.

The shape of the equipotential $U = 0$ is found by setting $\nu_2 - \nu_1 = 0$ and that of $V = 0$ by setting $\mu_2 - \mu_1 = 0$. Since $\tan^2 \nu_j$ (and hence $\tan^2 \mu_j$) is the root of an algebraic equation, the shape of these equipotentials is determined by algebraic expressions. This is a most interesting result considering that the expression for the potential function has the form of an incomplete elliptic integral.

Explicit expressions for $U = 0$ and $V = 0$, obtained by algebraic manipulation of Eqs. (20) and (17) are rather cumbersome and are not reproduced here.

To determine the shape of the other equipotentials, we use formula 116.01 [7]. Thus, e.g.,

$$U(z, r) = \frac{F(\nu_2, k) - F(\nu_1, k)}{\sqrt{p}} = \frac{F(\epsilon, k)}{\sqrt{p}} \tag{21}$$

where

$$\tan \frac{\epsilon}{2} = \frac{\sin \nu_2 \sqrt{1 - k^2 \sin^2 \nu_1} - \sin \nu_1 \sqrt{1 - k^2 \sin^2 \nu_2}}{\cos \nu_1 + \cos \nu_2},$$

The relation between ϵ and k for which $F(\epsilon, k)$ is constant is determined numerically. It is shown graphically in Fig. 4 [7]. With regard to this determination, it is to be noted that k is a function of z and r only and not of the limits of integration.

As shown by the examples above, the general nature of the boundaries is determined for every given kernel. This means that the boundary value problem is worked out backwards. We find the answer to the following question: Given a kernel, what boundary value problem can be solved? In order to obtain a solution to a physical problem, we must be able to preassign (to any desired degree of approximation) the shape of the boundary surfaces. One approach is to superimpose two or more solutions and see what conclusions can be drawn concerning the resulting boundaries and how much leeway there is in preassigning such boundaries. Instead of a kernel expansion, it may be more convenient to use a superposition of the form

$$W(z, r) = \sum_{n=0} C_n \int_{q_{2n+1}}^{q_{2n+2}} \frac{dt}{\sqrt{(t - z)^2 + r^2}}, \qquad q_{2n+j} = a_{2n+j} + ib_{2n+j} .$$

In subsequent papers of this series, we plan to extend the kernel technique to equations of the Laplace type with different symmetries and a larger number of variables.

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